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THE CANCELLATION PROBLEM IN DIMENSION FOUR

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The Cancellation Problem in Dimension Four

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Abstract

This paper proves that the Cancellation Problem has an affirmative answer over a Dedekind containing the rational numbers in dimension three. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of locally nilpotent derivations in dimension four, including the triangular ones.

1 Introduction

Let k be a field of characteristic zero and let V be an algebraic variety over k . The Cancellation Problem asks if $V \times k \cong k^n$ implies that $V \cong k^{n-1}$. This problem was first posed by Zariski in 1942. See [Kra89] for an overview of the Cancellation Problem.

Algebraically, the Cancellation Problem amounts to asking if $A[T] \cong k[X_1, \dots, X_n]$ implies that $A \cong k[X_1, \dots, X_{n-1}]$ for an affine k -domain A . One can also phrase this in terms of locally nilpotent derivations. The question is whether the kernel of a locally nilpotent derivation on $k[X_1, \dots, X_n]$ with a slice is isomorphic to $k[X_1, \dots, X_{n-1}]$. For more information on locally nilpotent derivations and their application to problems related to the Cancellation Problem, such as the Embedding Conjecture, Hilbert's Fourteenth Problem, and the Jacobian Conjecture, see [Ren68], [AM75], [Ess93], [Now94], [DF99], [Fre00] and [Ess00].

The structure of this paper is as follows. Section 2 contains an overview of locally nilpotent derivations and their relationship to the Cancellation Problem. Section 3 uses a technique from Quillen to show that local coordinates (and partial local coordinate systems) over a Hermite domain are coordinates (and partial coordinate systems). Together with a result from Sathaye on the recognition of a polynomial ring in two variables over a discrete valuation ring containing \mathbb{Q} , this result is used to prove that the Cancellation Problem has an affirmative answer over a Dedekind domain containing \mathbb{Q} for $n = 3$. As a consequence, the Cancellation Problem turns out to have an affirmative answer for a large class of derivations (or varieties) for $n = 4$ over a field, including the triangular derivations.

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2 Preliminaries

In this paper all rings will be commutative and have a unit element.

Let A be a ring. A *derivation* on A is a map $D: A \rightarrow A$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = aD(b) + D(a)b$ for all $a, b \in A$. If R is a ring and A is an R -algebra via $f: R \rightarrow A$, then A is called an R -derivation if $D(f(r)) = 0$ for all $r \in R$. A derivation D is called *locally nilpotent* if for all $a \in A$ there is an $n \in \mathbb{N}$ such that $D^n(a) = 0$. The kernel of D is denoted by A^D . If $s \in A$ is such that $D(s) = 1$, then s is called a *slice* of D .

The following proposition (see [Wri81]) is well-known.

Proposition 2.1. *Let A be a \mathbb{Q} -algebra and let D be a locally nilpotent derivation on A . Assume that $s \in A$ is a slice of D . Then $A = A^D[s]$ and s is algebraically independent over A^D . Furthermore, $D = d/ds$. \square*

In the applications, A will invariably be a polynomial ring $R[X] := R[X_1, \dots, X_n]$ over a ring R . An R -derivation D on such a ring is called *triangular* if $D(X_i) \in R[X_{i+1}, \dots, X_n]$ for all i . Such a derivation is automatically locally nilpotent. An element $s \in R[X]$ is called a *coordinate* if there is a polynomial automorphism F of $R[X]$ with s as one of its components. More generally, a sequence (s_1, \dots, s_k) of elements of $R[X]$ with $1 \leq k \leq n$ is called a *partial coordinate system* if there are polynomials $f_{k+1}, \dots, f_n \in R[X]$ such that $(s_1, \dots, s_k, f_{k+1}, \dots, f_n)$ is a polynomial automorphism of $R[X]$.

Proposition 2.1 implies the following.

Corollary 2.2. *Let R be a ring and let $n \geq 2$. Let D be a locally nilpotent R -derivation on $R[X] := R[X_1, \dots, X_n]$ and let $s \in R[X]$ be a slice of D . Then s is a coordinate if and only if $R[X]^D \cong R^{[n-1]}$. \square*

This gives the following reformulation of the Cancellation Problem in terms of locally nilpotent derivations.

Problem 2.3 (Cancellation Problem). *Let k be a field of characteristic zero and let $n \geq 2$. Let D be a locally nilpotent k -derivation on $k[X] := k[X_1, \dots, X_n]$ and assume that D has a slice $s \in k[X]$. Is then $k[X]^D \cong k^{[n-1]}$, i.e., is s a coordinate in $k[X]$?*

More generally, one can ask the following question.

Problem 2.4 (Generalized Cancellation Problem). *Let k be a field of characteristic zero, let R be an affine k -domain, and let $n \geq 2$. Let D be a locally nilpotent R -derivation on $R[X] := R[X_1, \dots, X_n]$ and assume that D has a slice $s \in R[X]$. Is then $R[X]^D \cong R^{[n-1]}$, i.e., is s a coordinate in $R[X]$?*

In dimension two, matters were settled for the field case by Rentschler in [Ren68], who proved the following.

Theorem 2.5. *Let k be a field of characteristic zero. Let D be a locally nilpotent k -derivation on $k[X] := k[X_1, X_2]$. Then $k[X]^D \cong k^{[1]}$. \square*

Nowadays even stronger results have been obtained by Bhadwadekar and Dutta ([BD97]) and Berson, Van den Essen, and Maubach ([BEM99]). The field k in the theorem can in fact be replaced by an arbitrary \mathbb{Q} -algebra R .

In dimension three, the Cancellation Problem was proved by Fujita (see [Fuj79]) for an algebraically closed field. See also [MS80] and [Miy85]. It was remarked by Daigle in [Dai97] that a straightforward use of [Kam75] then proves the general case.

Theorem 2.6. *Let k be a field of characteristic zero and let D be a locally nilpotent k -derivation on $k[X] := k[X_1, X_2, X_3]$. Assume that D has a slice. Then $k[X]^D \cong k^{[2]}$. \square*

This paper now proves the Generalized Cancellation Problem for $n = 3$ in case R is a Dedekind domain over \mathbb{Q} . As a consequence, the Cancellation Problem turns out to have an affirmative answer for locally nilpotent derivations of the form

$$D := a(X_1, X_2, X_3, X_4)\partial_1 + b(X_1, X_2, X_3, X_4)\partial_2 + c(X_1, X_2, X_3, X_4)\partial_3 + d(X_4)\partial_4$$

for $n = 4$, where ∂_i denotes $\partial/\partial X_i$. In particular, the Cancellation Problem turns out to have an affirmative answer for triangular derivations for $n = 4$. This is especially interesting since [Asa99] (implicitly) and [ER00] (explicitly) give a candidate counterexample to the Cancellation Problem for $n = 5$ which is triangular, namely $D := (2X_4^2 - 3)\partial_1 + (4X_4^3 - 8X_4)\partial_2 + (5X_4^4 - 10)\partial_3 + X_5\partial_4$.

3 Local Coordinates

Let R be a domain, $n \in \mathbb{N}$, and $R[X] := R[X_1, \dots, X_n]$ the polynomial ring in n variables over R . This section shows that a polynomial in $R[X]$ is a coordinate if and only if it is a coordinate when considered as an element of $R_{\mathfrak{m}}[X]$, for all maximal ideals \mathfrak{m} of R , provided that R is Hermite, and similarly for partial coordinate systems. Recall that R is called *Hermite* if every unimodular row (r_1, \dots, r_k) can be extended to an invertible square matrix over R .

The ideas present in this section can in fact already be found in [Qui76]. The abstract notion of Quillen Induction is essentially taken from [BCW77] and the results from that paper can also be used to derive the main result of this section.

Definition 3.1. Define $\text{Loc}(R) := \{R_r \mid r \in R \setminus \{0\}\}$.

Proposition 3.2 (Quillen Induction). *Let $P \subseteq \text{Loc}(R)$. Write $P(L)$ instead of $L \in P$ for $L \in \text{Loc}(R)$. In that case, L is said to have property P . Assume that*

- (a) *for all $\mathfrak{m} \in \text{Max}(R)$: there exists an $r \in R \setminus \mathfrak{m}$ such that $P(R_r)$;*
- (b) *for all $r, s, t \in R \setminus \{0\}$: if $rR_t + sR_t = R_t$, $P(R_r)$, and $P(R_s)$, then $P(R_t)$.*

Then $P(L)$ for all $L \in \text{Loc}(R)$. In particular $P(R)$.

Proof. Let S be the collection of all $r \in R \setminus \{0\}$ such that $P(R_r)$ together with 0. This is an ideal of R . It is not empty because $0 \in S$, closed under addition because of (b) (for $r, s \in S$ take $t := r + s$), and closed under multiplication with elements of R also because of (b) (for $\tilde{r} \in R$ and $\tilde{s} \in S$, take $r := \tilde{s}$, $s := \tilde{s}$, and $t := \tilde{r}\tilde{s}$).

Suppose that $S \neq R$. Then S is contained in some maximal ideal of R , say \mathfrak{m} . By (a) there is an $r \in R \setminus \mathfrak{m}$ such that $P(R_r)$. But then $r \in S \subseteq \mathfrak{m}$, which contradicts $r \notin \mathfrak{m}$. So $S = R$ and therefore $P(L)$ for all $L \in \text{Loc}(R)$. \square

Definition 3.3. An element H of $\text{End}_R(R[X])$ is called *nice* if it is of the form $H = (X_1 + \text{h.o.t.}, \dots, X_n + \text{h.o.t.})$. Here h.o.t. stands for higher order terms, *i.e.*, terms of degree 2 or greater, and $\text{End}_R(R[X])$ has been identified with $R[X]^n$. A coordinate $h \in R[X]$ is called *nice* if there is a nice $H \in \text{Aut}_R(R[X])$ which has h as its first component. Similarly, a partial coordinate system $(h_1, \dots, h_k) \in R[X]^k$ is called *nice* if there is a nice $H \in \text{Aut}_R(R[X])$ which has (h_1, \dots, h_k) as its first k components.

Lemma 3.4. A partial coordinate system $(h_1, \dots, h_k) \in R[X]^k$ is nice if and only if it is of the form $(X_1 + \text{h.o.t.}, \dots, X_k + \text{h.o.t.})$. In particular, a coordinate $h \in R[X]$ is nice if and only if it is of the form $X_1 + \text{h.o.t.}$.

Proof. By linear algebra, looking at the linear part of a polynomial automorphism without constant parts with h_1, \dots, h_k as its first k components. \square

Definition 3.5. Let $H \in \text{End}_R(R[X])$ be nice. Then ${}^T H \in \text{End}_{R[T]}(R[T][X])$ is defined by

$${}^T H := T^{-1}H[X_1 := TX_1, \dots, X_n := TX_n].$$

(This is defined over $R[T]$ and not just over $R[T, T^{-1}]$ because H is nice.) If $r \in R$, then ${}^T H[T := r] \in \text{End}_R(R[X])$ is denoted by ${}^r H$.

One can easily see that $(\det JH)[X := TX] = \det J{}^T H$ and that H is invertible if and only if ${}^T H$ is. Here JH denotes the Jacobian matrix $(\partial H_i / \partial X_j)_{ij}$ of H . Even better, if $r \in R \setminus \{0\}$, then $\det J{}^r H \in R^*$ if and only if $\det JH \in R^*$ and ${}^r H$ is invertible if and only if H is.

The map ${}^T H$ is called the *clearing map* because of the following: if K is the quotient field of R and $H \in \text{End}_K(K[X])$ is of the form $H = X + \text{h.o.t.}$, then there is an $r \in R \setminus \{0\}$ such that ${}^r H \in \text{End}_R(R[X])$. So, the denominators of H are cleared. See Chapter 1 of [Ess00].

Lemma 3.6. Let $r, s \in R \setminus \{0\}$ be such that $rR + sR = R$ and let $H \in \text{Aut}_{R_{rs}}(R_{rs}[X])$ be nice. Then there are nice $H_1 \in \text{Aut}_{R_r}(R_r[X])$ and $H_2 \in \text{Aut}_{R_s}(R_s[X])$ such that $H = H_1 H_2$.

Proof. Note that

$${}^T H = H_{(1)} + T H_{(2)} + T^2 H_{(3)} + \dots + T^{d-1} H_{(d)}$$

where each $H_{(i)}$ is the homogeneous part of degree i of H and d is the degree of H . Hence

$$\begin{aligned} {}^{1-T}H &= H_{(1)} + (1-T)H_{(2)} + (1-T)^2H_{(3)} + \cdots + (1-T)^{d-1}H_{(d)} \\ &= H_{(1)} + H_{(2)} + H_{(3)} + \cdots + H_{(d)} + T(\text{h.o.t.}) \\ &= H + T(\text{h.o.t.}), \end{aligned}$$

where, as before, h.o.t. stands for some terms of X -degree at least two. As a consequence

$$\begin{aligned} H^{-1} \circ {}^{1-T}H &= H^{-1} \circ (H + T(\text{h.o.t.})) \\ &= X + T(\text{h.o.t.}). \end{aligned}$$

Now let $k \in \mathbb{N}$ be sufficiently large. From $rR + sR = R$ it follows that $r^kR + s^kR = R$. Take $v, w \in R$ with $r^kv + s^kw = 1$. If k is sufficiently large, then ${}^{s^kw}H$ and ${}^{s^kw}(H^{-1})$ are elements of $\text{End}_{R_r}(R_r[X])$. They are also each others inverse and hence they are in fact elements of $\text{Aut}_{R_r}(R_r[X])$.

Take $H_1 := {}^{s^kw}H$ and compute $H^{-1}H_1$. This gives

$$\begin{aligned} H^{-1}H_1 &= H^{-1} \circ {}^TH [T := s^kw] \\ &= H^{-1} \circ {}^{1-T}H [T := r^kv] \\ &= (X + T(\text{h.o.t.})) [T := r^kv] \\ &= X + r^kv(\text{h.o.t.}) \end{aligned}$$

and similarly

$$H_1^{-1}H = X + r^kv(\text{h.o.t.}).$$

For k sufficiently large, $H_2 := H_1^{-1}H$ and its inverse apparently are elements of $\text{Aut}_{R_s}(R_s[X])$. So now $H = H_1H_2$ with H_1 and H_2 are both of the required form. \square

Lemma 3.7. *Let $r, s \in R$ be such that $rR + sR = R$. Take $t \in R_{rs}$ such that $t \in R_r \cap R_s$. Then $t \in R$.*

Proof. Write $t = v/r^k = w/s^l$ with $v, w \in R$ and $k, l \in \mathbb{N}$. Because $rR + sR = R$, also $r^kR + s^lR = R$. Write $r^kx + s^ly = 1$ for some $x, y \in R$. Then $t = (r^kx + s^ly)t = vx + wy \in R$. \square

Lemma 3.8 (Patching Lemma). *Let $r, s \in R$ with $rR + sR = R$. Let $k \in \{1, \dots, n\}$ and let $h_1, \dots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that there is a nice $F \in \text{Aut}_{R_r}(R_r[X])$ with first k components equal to h_1, \dots, h_k and that there is a nice $G \in \text{Aut}_{R_s}(R_s[X])$ with first k components equal to h_1, \dots, h_k . Then there is a nice $H \in \text{Aut}_R(R[X])$ with first k components equal to h_1, \dots, h_k .*

Proof. Consider the polynomial map $F^{-1}G \in \text{Aut}_{R_{rs}}(R_{rs}[X])$ and note that it is fact an $R_{rs}[X_1, \dots, X_k]$ -automorphism of $R_{rs}[X] = R_{rs}[X_1, \dots, X_k][X_{k+1}, \dots, X_n]$. Now apply Lemma 3.6 to the ring $R[X_1, \dots, X_k]$ and write $F^{-1}G = H_1H_2$ with $H_1 \in \text{Aut}_{R_r[X_1, \dots, X_k]}(R_r[X])$ and $H_2 \in \text{Aut}_{R_s[X_1, \dots, X_k]}(R_s[X])$, where both H_i are of the form $X + \text{h.o.t.}$. Considered as automorphisms over respectively R_r and R_s , the first k components of H_1 and H_2 ofcourse equal X_1, \dots, X_k . Hence $H := FH_1 = GH_2^{-1}$ is a nice polynomial automorphism (over R_{rs} , a priori) whose first k components equal h_1, \dots, h_k . It is defined over R_r (because $H = FH_1$ and F and H_1 are defined over R_r) and it is defined over R_s (because $H = GH_2^{-1}$ and G and H_2 are defined over R_s). Hence, applying Lemma 3.7 to every one of its coefficients, it is in fact defined over R . \square

Theorem 3.9. *Let $k \in \{1, \dots, n\}$ and let $h_1, \dots, h_k \in R[X]$ be polynomials of the form $h_i = X_i + \text{h.o.t.}$. Assume that for every maximal ideal \mathfrak{m} of R , (h_1, \dots, h_k) is a nice partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$. Then (h_1, \dots, h_k) is a nice partial coordinate system.*

Proof. Let $P \subseteq \text{Loc}(R)$ be the collection of all R_r , $r \in R \setminus \{0\}$, such that (h_1, \dots, h_k) is a nice partial coordinate system over R_r . Now check the two conditions for Quillen Induction.

- (a) Let \mathfrak{m} be a maximal ideal of R . It is assumed that (h_1, \dots, h_k) is a nice partial coordinate system over $R_{\mathfrak{m}}$. Using Lemma 3.4, choose $F \in \text{Aut}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$ nice with first k components equal to h_1, \dots, h_k . There are only finitely many elements of R appearing in the denominator of a coefficient of a component of F and its inverse. Denote the product of these denominators by r . None of these denominators is an element of \mathfrak{m} and, because \mathfrak{m} is prime, r is not an element of \mathfrak{m} either. Furthermore, obviously, $P(R_r)$.
- (b) Let $r, s, t \in R \setminus \{0\}$ be such that $rR_t + sR_t = R_t$ and assume $P(R_r)$ and $P(R_s)$. Then $P(R_t)$ follows by applying the Patching Lemma (Lemma 3.8) to the ring R_t .

So, using Quillen Induction (Proposition 3.2), $P(R)$, which means that (h_1, \dots, h_k) is a nice partial coordinate system over R . \square

Corollary 3.10. *Assume that R is Hermite. Let $k \in \{1, \dots, n\}$ and $h_1, \dots, h_k \in R[X]$. Assume that (h_1, \dots, h_k) is a partial coordinate system when considered as an element of $R_{\mathfrak{m}}[X]^k$, for every maximal ideal \mathfrak{m} of R . Then (h_1, \dots, h_k) is a partial coordinate system.*

Proof. First of all note that it is possible to assume that the h_i have no constant part. Write $h_i = r_{i1}X_1 + \dots + r_{in}X_n + \text{h.o.t.}$ for all i , with $r_{ij} \in R$.

Consider a maximal ideal \mathfrak{m} of R . Then (h_1, \dots, h_k) is a partial coordinate system over $R_{\mathfrak{m}}$, which means that there are $f_{k+1}, \dots, f_n \in R_{\mathfrak{m}}[X]$ such that $F := (h_1, \dots, h_k, f_{k+1}, \dots, f_n) \in \text{Aut}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}[X])$. The f_i can be chosen in such a way

that they have no constant part. Then $\det JF \in R_{\mathfrak{m}}[X]^*$ and hence substituting $X_1 := 0, \dots, X_n := 0$ gives

$$\begin{vmatrix} r_{11} & \dots & r_{1n} \\ \vdots & & \vdots \\ r_{k1} & \dots & r_{kn} \\ * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{vmatrix} = \det J(F[X := 0]) = (\det JF)[X := 0] \in R_{\mathfrak{m}}^*.$$

In particular, the matrix $(r_{ij})_{ij}$ represents a surjective $R_{\mathfrak{m}}$ -module homomorphism from $R_{\mathfrak{m}}^n$ to $R_{\mathfrak{m}}^k$.

Because this holds for every maximal ideal of R , it follows that the matrix $(r_{ij})_{ij}$ represents a surjective R -module homomorphism from R^n to R^k . Now R is Hermite, which implies that the matrix $(r_{ij})_{ij}$ can be extended to an invertible square matrix M over R (see [Lam78], Corollary 4.5). Viewing this matrix M as a polynomial automorphism of $R[X]$ and applying its inverse to the polynomials h_i , it follows that one can assume that (h_1, \dots, h_k) is of the form $(X_1 + \text{h.o.t.}, \dots, X_k + \text{h.o.t.})$. By Lemma 3.4, (h_1, \dots, h_k) then is a nice coordinate system in $R_{\mathfrak{m}}[X]$, for every $\mathfrak{m} \in \text{Max}(R)$. Now apply Theorem 3.9. \square

The condition that R be Hermite in the previous corollary is necessary. For let R be any non-Hermite ring; say (a_1, \dots, a_n) is a unimodular row over R that cannot be extended to an invertible square matrix. Then $h := a_1 X_1 + \dots + a_n X_n \in R[X_1, \dots, X_n]$ is not a coordinate (if it were, the coefficients of the linear part of an automorphism with h as its first component would form an invertible square matrix over R extending (a_1, \dots, a_n)). However, localising in a maximal ideal \mathfrak{m} of R , (a_1, \dots, a_n) is extendible to an invertible square matrix over $R_{\mathfrak{m}}$ (since $R_{\mathfrak{m}}$ is local) and so h is a coordinate over $R_{\mathfrak{m}}$.

4 Main Result

In [Sat83], Sathaye proved the following characterization of a polynomial ring in two variables over an discrete valuation ring containing \mathbb{Q} .

Theorem 4.1. *Let R be a discrete valuation ring containing \mathbb{Q} . Denote the unique maximal ideal of R by \mathfrak{m} , write K for the quotient field $Q(R)$ of R , and write k for the residue field R/\mathfrak{m} of R . Let A be a finitely generated affine R -domain and assume that $K \otimes_R A \cong K^{[2]}$ and that $k \otimes_R A \cong k^{[2]}$. Then $A \cong R^{[2]}$.* \square

In order to use this result, a lemma is needed on the behaviour of the kernel of a locally nilpotent derivation with a slice under tensoring.

Lemma 4.2. *Let $s \in R[X] := R[X_1, \dots, X_n]$ and let A be an R -algebra via the map $\varphi: R \rightarrow A$. Denote the induced map $R[X] \rightarrow A[X]$ by $\varphi_{\#}$. Then*

$$A \otimes_R R[X]/(sR[X]) \cong A[X]/(\varphi_{\#}(s)A[X])$$

In particular, if D is a locally nilpotent R -derivation on $R[X]$ and s is a slice of D , then

$$A \otimes_R R[X]^D \cong A[X]^{\tilde{D}},$$

where \tilde{D} denotes the extension of D to $A[X]$.

Proof. The following diagram is a commutative diagram of R -modules and R -module homomorphism in which the horizontal sequences are exact.

$$\begin{array}{ccccccc} sR[X] & \longrightarrow & R[X] & \longrightarrow & R[X]/sR[X] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A \otimes_R sR[X] & \longrightarrow & A \otimes_R R[X] & \longrightarrow & A \otimes_R R[X]/sR[X] & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \varphi_{\#}(s)A[X] & \longrightarrow & A[X] & \longrightarrow & A[X]/(\varphi_{\#}(s)A[X]) & \longrightarrow & 0 \end{array}$$

The map $A \otimes_R sR[X] \rightarrow \varphi_{\#}(s)A[X]$ is surjective: take an element $\varphi_{\#}(s)f \in A[X]$ with $f \in A[X]$. Write $f = \sum_{\alpha} c_{\alpha} X_1^{\alpha_1} \dots X_n^{\alpha_n}$ with each $c_{\alpha} \in A$. Then $\varphi_{\#}(s)f$ is the image of $\sum_{\alpha} c_{\alpha} \otimes sX_1^{\alpha_1} \dots X_n^{\alpha_n}$. Also, the map $A \otimes_R R[X] \rightarrow A[X]$ is an isomorphism. Hence, by the Five Lemma, the map $A \otimes_R R[X]/sR[X] \rightarrow A[X]/(\varphi_{\#}(s)A[X])$ is an isomorphism. A priori this is an isomorphism of R -modules. However, since it is an A -module homomorphism, it is even an isomorphism of A -modules.

The second claim follows from the first one using Theorem 2.1. \square

Note that this lemma is false if D does not have slice. For instance, let K be some field, $R := K[Y]$, and consider $A := K$ as an R -module by sending elements of K to themselves and Y to 0. Let D be the locally nilpotent derivation $Y\partial_X$ on $R[X]$. Then $R[X]^D = R$, so $A \otimes_R R[X]^D = A = K$. However, the extension \tilde{D} of D to $A[X]$ is 0 and hence $A[X]^{\tilde{D}} = A[X]$.

Lemma 4.3. *Let R be a discrete valuation ring containing \mathbb{Q} and let D be a locally nilpotent R -derivation on $R[X, Y, Z]$ with a slice $s \in R[X, Y, Z]$. Then $R[X, Y, Z]^D \cong R^{[2]}$.*

Proof. Let k be the residue field of R and let K be the quotient field of R . Denote the extension of D to $K \otimes_R R[X, Y, Z] \cong K[X, Y, Z]$ by \tilde{D} . By Lemma 4.2 and Theorem 2.6 it follows that

$$K \otimes_R R[X, Y, Z]^D \cong K[X, Y, Z]^{\tilde{D}} \cong K^{[2]}.$$

In exactly the same way it follows that

$$k \otimes_R R[X, Y, Z]^D \cong k^{[2]}.$$

Hence, by Theorem 4.1, $R[X, Y, Z]^D \cong R^{[2]}$. \square

Theorem 4.4. *Let R be a Dedekind domain containing \mathbb{Q} and let D be a locally nilpotent R -derivation on $R[X, Y, Z]$ with a slice. Then $R[X, Y, Z]^D \cong R^{[2]}$.*

Proof. Let $s \in R[X, Y, Z]$ be a slice of D . Note that a unimodular row of length 2 is always extendible to an invertible square matrix and by Bass' Cancellation Theorem for Stably Free Modules ([Bas68], Theorem V.3.2; see also [Wei00], Theorem 1.3) every unimodular row of length at least 3 over a Noetherian ring of dimension one is extendible. In particular, R is Hermite. By Corollary 3.10 it is enough to show that s is a coordinate in $R_{\mathfrak{m}}[X, Y, Z]$ for every maximal ideal \mathfrak{m} of R .

So let \mathfrak{m} be a maximal ideal of R . Then $R_{\mathfrak{m}}$ is a discrete valuation ring. Because R contains \mathbb{Q} , $R_{\mathfrak{m}}$ contains \mathbb{Q} as well. Now Lemma 4.3 implies that $R_{\mathfrak{m}}[X, Y, Z]^D \cong R_{\mathfrak{m}}^{[2]}$. In other words, s is a coordinate in $R_{\mathfrak{m}}[X, Y, Z]$. \square

Corollary 4.5. *Let k be a field of characteristic zero and let D be a locally nilpotent k -derivation on $k[X, Y, Z, W]$ of the form*

$$D := a(X, Y, Z, W)\partial_X + b(X, Y, Z, W)\partial_Y + c(X, Y, Z, W)\partial_Z + d(W)\partial_W.$$

Assume that D has a slice. Then $k[X, Y, Z, W]^D \cong k^{[3]}$.

Proof. If $d(W) \neq 0$, then $d(W) \in k^*$, since D is locally nilpotent. So $d^{-1}W$ is a slice of D . This slice is also a coordinate and hence $k[X]^D \cong k^{[3]}$. Otherwise, if $d(W) = 0$, apply Theorem 4.4 with $R = k[W]$. \square

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